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Generalized integrable hierarchies and Combescure symmetry transformations

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Abstract. Unifying hierarchies of integrable equations are discussed. They are constructed via the generalized Hirota identity. It is shown that the Combescure transformations, known for a long time for the Darboux system and having a simple geometrical meaning, are in fact the symmetry transformations of generalized integrable hierarchies, though the connection with geometry in the general case is not clear. The generalized equation written in terms of invariants of Combescure transformations are the usual integrable equations and their modified partners. The KP–mKP, DS–mDS hierarchies and Darboux system are considered.

1. Introduction

The Sato approach (see e.g. [1–3]) and the $\bar{\partial}$-dressing method (see e.g. [4–7]) are two powerful tools with which to construct and analyse the hierarchies of integrable equations. A bridge between these seemingly different approaches has been established by the observation that the Hirota bilinear identity can be derived from the $\bar{\partial}$-equation [7, 8]. An approach which combines the characteristic features of both methods, namely, the Hirota bilinear identity from the Sato approach and the analytic properties of solutions from the $\bar{\partial}$-dressing method, has been discussed in [9–11].

A connection between wavefunctions with different normalizations was one of interesting open problems of the $\bar{\partial}$-dressing method. In [9] it was shown that such a connection is given by the Combescure transformation.

The Combescure transformation was introduced during the last century within the study of the transformation properties of surfaces (see e.g. [12, 13]). It is a transformation of the surface such that all the tangent vectors at a given point of the surface remain parallel. The Combescure transformation is essentially different from the well known Bäcklund and Darboux transformations. The Combescure transformation plays an important role in the theory of systems of hydrodynamical type [14]. It is also of great interest for the theory of $(2+1)$-dimensional integrable systems [15].

The primary motivation of this work is to interpret the geometrical Combescure transformation for the triply-conjugate systems of surfaces in three-dimensional space [12], described by the scalar case of the integrable Darboux–Zakharov–Manakov (DZM) system

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[4], in the context of integrable systems (in the frame of the $\bar{\partial}$-dressing method, see [9]). It appears that in this context the meaning of the Combescure transformation is very transparent, and broad classes of integrable equations possess such a type of symmetry transformation, though a geometrical sense of it in the general case is not clear. Nevertheless, we call this type of symmetry transformation *a Combescure transformation*.

The main technical feature (which we believe is new) of our approach to integrable hierarchies is the consistent use of the wavefunction $\psi(\lambda, \mu)$ with simple analytic properties (the Cauchy–Baker–Akhiezer (CBA) function). We derive generalized integrable hierarchies in terms of the function $\psi(\lambda, \mu)$, starting from the generalized Hirota bilinear identity for this function. Such generalized hierarchies contain the usual integrable equations, their modified partners and corresponding linear problems. The compact form of the generalized equations is derived in terms of the τ-function. This approach provides us with a natural framework with which to interpret the Combescure transformation for the DZM system and to transfer the notion of the Combescure symmetry transformation to generalized hierarchies of integrable equations.

It is shown that the generalized equations possess the symmetries given by the Combescure transformations. The invariants of these symmetry transformations are found. The generalized equations written in terms of these invariants coincide with the usual equations or their modified versions. The Darboux transformation and its connection with the Combescure transformation for the generalized hierarchies is also discussed. The Kadomtsev–Petviashvili (KP) and modified KP (mKP) equations, the Davey–Stewartson (DS) and modified Davey–Stewartson (mDS) equations and the matrix Darboux–Zakharov–Manakov (DZM) system are considered.

It is interesting to note that in the case of the generalized KP–mKP hierarchy the *singularity manifold equation* naturally appears as an equation posessing a full Combescure symmetry group. This equation first arose in a completely different context in the Painleve analysis of the KP equation [16]. The invariant of this equation under the full Combescure group is described by the KP equation, while the invariants under the action of its right or left subgroups (see later) are described by the mKP or dual mKP equations. So in our approach the KP hierarchy, the mKP hierarchy and the singularity manifold equation hierarchy are united into the *generalized KP hierarchy*, while the connection between the different levels of the hierarchy is described in terms of the invariants of the Combescure symmetry transformations group. So the Combescure symmetry transformations group plays a fundamental role in the structure of integrable hierarchies.

2. Generalized Hirota identity

As mentioned in section 1, the important feature of our approach is the consistent use of the CBA function. We do not give a constructive definition of this function here, but it is correctly defined in frame of the $\bar{\partial}$-dressing method [9]; in the frame of the algebro-geometric technique the function $\psi(\lambda, \mu)$ corresponds to Cauchy–Baker–Akhiezer kernel on the Riemann surface (see [17]), so the equations obtained by our approach are consistent (they possess a broad class of solutions). We believe that the notion of the CBA function can be introduced to the Segal–Wilson Grassmannian approach [3] and to the Sato approach [1], some hints in this direction can be found in [10] and [11]. In some sense the CBA function is a special basic function generating two vector spaces corresponding to a Grassmannian and dual Grassmannian point.

In the present work we introduce from the beginning the generalized Hirota bilinear identity for the CBA function, and using this identity we derive generalized integrable hierarchies and their symmetry transformations. In fact every formula contained in this work is derived from the generalized Hirota bilinear identity, so in this sense this work is closed and self-consistent and requires no extra technique. Taking into account the importance of this identity for our work, we treat it here in some detail (see also [11]).

Thus we start with the generalized Hirota bilinear identity

$$\int_{\partial G} \chi(\nu, \mu; g_1) g_1(\nu) g_2^{1}(\nu) \chi(\lambda, \nu; g_2)\, \mathrm{d}\nu = 0. \tag{1}$$

Here $\chi(\lambda, \mu; g)$ is a matrix function of two complex variables $\lambda, \mu \in G$ and a functional of the group element g defining the dynamics (which will be specified later), G is some set of domains of the complex plane, the integration goes over the boundary of G. By definition, the function $\chi(\lambda, \mu)$ possesses the following analytical properties:

$$\bar{\partial}_\lambda \chi(\lambda, \mu) = 2\pi \mathrm{i} \delta(\lambda - \mu) \qquad -\bar{\partial}_\mu \chi(\lambda, \mu) = 2\pi \mathrm{i} \delta(\lambda - \mu)$$

where $\delta(\lambda - \mu)$ is a δ-function, or, in other words, $\chi \to (\lambda - \mu)^{-1}$ as $\lambda \to \mu$ and $\chi(\lambda, \mu)$ is an analytic function of both variables λ, μ for $\lambda \neq \mu$.

The formula (1) is a basic tool of our construction.

We suggest in what follows that we are able to find solutions to it somehow; we treat different constructive methods as methods with which to find solutions to the generalized Hirota bilinear identity.

In fact some special solutions (determinant solutions and degenerate solutions) to this identity can be found directly, without any additional construction (the CBA function for $g = 1$ plays the role of the initial data).

In another form, more similar to the standard Hirota bilinear identity, the identity (1) can be written as

$$\int_{\partial G} \psi(\nu, \mu; g_1) \psi(\lambda, \nu; g_2)\, \mathrm{d}\nu = 0 \tag{2}$$

where

$$\psi(\lambda, \mu; g) = g^{1}(\mu) \chi(\lambda, \mu; g) g(\lambda). \tag{3}$$

We call the function $\psi(\lambda, \mu; g)$ a *Cauchy–Baker–Akhiezer (CBA) function*.

Let us consider two linear spaces $W(g)$ and $\tilde{W}(g)$ defined by the function $\chi(\lambda, \mu)$ (satisfying (1)) via equations connected with the identity (1)

$$\int_{\partial G} f(\nu; g) \chi(\lambda, \nu; g)\, \mathrm{d}\nu = 0 \tag{4}$$

$$\int_{\partial G} \chi(\nu, \mu; g) h(\nu; g)\, \mathrm{d}\nu = 0 \tag{5}$$

here $f(\lambda) \in W$, $h(\lambda) \in \tilde{W}$; $f(\lambda)$, $h(\lambda)$ are defined in $\bar{G}$.

It follows from the definition of linear spaces W, $\tilde{W}$ that

$$\begin{aligned} f(\lambda) &= -2\pi \mathrm{i} \iint_G \eta(\nu) \chi(\lambda, \nu)\, \mathrm{d}\nu \wedge \mathrm{d}\bar{\nu} \qquad \eta(\nu) = \left(\frac{\partial}{\partial \bar{\nu}} f(\nu) \right) \\ h(\mu) &= 2\pi \mathrm{i} \iint_G \chi(\nu, \mu) \tilde{\eta}(\nu)\, \mathrm{d}\nu \wedge \mathrm{d}\bar{\nu} \qquad \tilde{\eta}(\nu) = \left(\frac{\partial}{\partial \bar{\nu}} h(\nu) \right). \end{aligned} \tag{6}$$

These formulae in some sense provide an expansion of the functions f, h in terms of the basic function $\chi(\lambda, \mu)$. The formulae (6) readily imply that linear spaces W, $\tilde{W}$ are transversal to the space of holomorphic functions in G (transversality property).

From the other point of view, these formulae define a map of the space of functions (distributions) on $\bar{G}$ η, $\tilde{\eta}$ to the spaces W, $\tilde{W}$. We will call η ($\tilde{\eta}$) a *normalization* of the corresponding function belonging to W ($\tilde{W}$).

The dynamics of the linear spaces W, $\tilde{W}$ looks very simple

$$W(g) = W_0 g^{-1} \qquad \tilde{W}(g) = g\tilde{W}_0. \tag{7}$$

Here $W_0 = W(g=1)$, $\tilde{W}_0 = \tilde{W}(g=1)$ (the formulae (7) follow from the identity (1) and the formulae (6)).

A dependence of the function $\chi(\lambda,\mu)$ on dynamical variables is hidden in the function $g(\lambda)$. We will consider here only the case of continuous variables, for which

$$g_i = \exp(K_i x_i) \qquad \frac{\partial}{\partial x_i} g_i = K_i g_i. \tag{8}$$

Here $K_i(\lambda)$ are commuting matrix meromorphic functions.

To introduce a dependence on several variables (which may be of different type), one should consider a product of corresponding functions $g_i(\lambda)$ (all of them commute).

Let G be a unit disc and x_ν a variable corresponding to $K_\nu(\lambda) = A_\nu/(\lambda-\nu)$. Differentiating the identity (1) over x_ν, one obtains

$$\frac{A_\nu}{\lambda-\nu}\frac{\partial}{\partial x_\nu}\chi(\lambda,\mu,x_\nu) - \frac{\partial}{\partial x_\nu}\chi(\lambda,\mu,x_\nu)\frac{A_\nu}{\mu-\nu} = \chi(\nu,\mu)A_\nu\chi(\lambda,\nu)$$

or, in terms of the ψ function (3),

$$\frac{\partial}{\partial x_\nu}\psi(\lambda,\mu,x_\nu) = \psi(\nu,\mu,x_\nu)A_\nu\psi(\lambda,\nu,x_\nu). \tag{9}$$

This formula allows one to construct the basic function $\chi(\lambda,\mu)$ using only two functions with 'canonical' normalization, the Baker–Akhiezer function $\psi(\lambda,\nu,x_\nu)$ and the dual Baker–Akhiezer function $\psi(\nu,\mu,x_\nu)$ corresponding to some fixed point ν.

3. The matrix DZM system

The matrix DZM system is our first example. In this case the construction and all formulae are very simple and transparent.

To derive the DZM system of equations, we take a set of three identical unit discs with the centre at $\lambda = 0$ $\boldsymbol{D}_i$, $1 \leqslant i \leqslant 3$, as G, and we denote the zero point of the corresponding disc as 0_i. The functions $K_i(\lambda)$, $1 \leqslant i \leqslant 3$, are chosen in the form

$$K_i(\lambda) = \frac{A_i}{\lambda} \qquad (\lambda \in \boldsymbol{D}_i)$$
$$K_i(\lambda) = 0 \qquad (\lambda \notin \boldsymbol{D}_i)$$

where A_i, A_j, A_k are commuting matrices.

It appears that the Hirota bilinear identity in differential form (9) contains enough information to derive equations for the rotation coefficients, DZM equations and the linear problem for the DZM equations. Indeed, evaluating the set of three relations (9) for independent variables x_i, x_j, x_k, $i \neq j \neq k \neq i$, at the set of points $\lambda,\mu \in \{0_i, 0_j, 0_k\}$, where, according to our notations, 0_i is the zero point of the disc $\boldsymbol{D}_i$ one easily obtains the relations

$$\partial_i\psi(\lambda,\mu,\boldsymbol{x}) = \psi(0_i,\mu,\boldsymbol{x})A_i\psi(\lambda,0_i,\boldsymbol{x})$$
$$\partial_i\psi(\lambda,0_j,\boldsymbol{x}) = \psi(0_i,0_j,\boldsymbol{x})A_i\psi(\lambda,0_i,\boldsymbol{x})$$
$$\partial_i\psi(0_j,\mu,\boldsymbol{x}) = \psi(0_i,\mu,\boldsymbol{x})A_i\psi(0_i,0_j,\boldsymbol{x})$$
$$\partial_i\psi(0_j,0_k,\boldsymbol{x}) = \psi(0_j,0_i,\boldsymbol{x})A_i\psi(0_i,0_k,\boldsymbol{x})$$

where $\partial_i = \partial/\partial x_i$, $\boldsymbol{x}$ is the set x_i, x_j, x_k.

Let us now take into account that all the equations containing λ, μ, can be integrated over the boundary of G with some matrix weight functions $\rho(\lambda)$, $\tilde{\rho}(\mu)$ (note that they are connected with the normalization functions defined by (6)) without changing the structure of equations. So we will write the equations in terms of *wavefunctions* independent of spectral parameters

$$\partial_i \Phi(\boldsymbol{x}) = \tilde{f}_i(\boldsymbol{x}) f_i(\boldsymbol{x}) \tag{10}$$

$$\partial_i f_j(\boldsymbol{x}) = \beta_{ji} f_i(\boldsymbol{x}) \tag{11}$$

$$\partial_i \tilde{f}_j(\boldsymbol{x}) = \tilde{f}_i(\boldsymbol{x}) \beta ij(\boldsymbol{x}) \tag{12}$$

$$\partial_i \beta_{jk}(\boldsymbol{x}) = \beta_{ji}(\boldsymbol{x}) \beta_{ik}(\boldsymbol{x}). \tag{13}$$

Here

$$\begin{aligned}
\Phi(\boldsymbol{x}) &= \iint \tilde{\rho}(\mu)\psi(\lambda, \mu, \boldsymbol{x})\rho(\lambda)\, \mathrm{d}\lambda\, \mathrm{d}\mu \\
f_i(\boldsymbol{x}) &= (A_i)^{\frac{1}{2}} \int \psi(\lambda, 0_i, \boldsymbol{x})\rho(\lambda)\, \mathrm{d}\lambda \\
\tilde{f}_i(\boldsymbol{x}) &= \int \tilde{\rho}(\mu)\psi(0_i, \mu, \boldsymbol{x})(A_i)^{\frac{1}{2}}\, \mathrm{d}\mu \\
\beta_{ij}(\boldsymbol{x}) &= (A_j)^{\frac{1}{2}} \psi(0_j, 0_i, \boldsymbol{x})(A_i)^{\frac{1}{2}}.
\end{aligned} \tag{14}$$

The system of equations (10)–(13) implies that

$$\partial_i \partial_j \tilde{f}_k(\boldsymbol{x}) = ((\partial_j \tilde{f}_i(\boldsymbol{x}))\tilde{f}_i(\boldsymbol{x})^{-1})\partial_i \tilde{f}_k(\boldsymbol{x}) + ((\partial_i \tilde{f}_j(\boldsymbol{x}))\tilde{f}_j(\boldsymbol{x})^{-1})\partial_j \tilde{f}_k(\boldsymbol{x}) \tag{15}$$

$$\partial_i \partial_j \Phi(\boldsymbol{x}) = ((\partial_j \tilde{f}_i(\boldsymbol{x}))\tilde{f}_i(\boldsymbol{x})^{-1})\partial_i \Phi(\boldsymbol{x}) + ((\partial_i \tilde{f}_j(\boldsymbol{x}))\tilde{f}_j(\boldsymbol{x})^{-1})\partial_j \Phi(\boldsymbol{x}) \tag{16}$$

and

$$\partial_i \partial_j f_k(\boldsymbol{x}) = (\partial_i f_k(\boldsymbol{x}))(f_i(\boldsymbol{x})^1 \partial_j f_i(\boldsymbol{x})) + (\partial_j f_k(\boldsymbol{x}))(f_j(\boldsymbol{x})^1 \partial_i f_j(\boldsymbol{x})) \tag{17}$$

$$\partial_i \partial_j \Phi(\boldsymbol{x}) = \partial_i \Phi(\boldsymbol{x}) f_i(\boldsymbol{x})^{-1}(\partial_j f_i(\boldsymbol{x})) + \partial_j \Phi(\boldsymbol{x}) f_j(\boldsymbol{x})^{-1}(\partial_i f_j(\boldsymbol{x})). \tag{18}$$

The system (15) is just the matrix DZM equation derived in [4]. The system (17) is its dual partner. So the solution for the DZM equations is, in fact, given by the *dual wavefunctions*, i.e. the wavefunctions for the linear equations (12), while the compatibility conditions for these equations give the equations for rotation coefficients (13). Solutions of the dual DZM system are given by the wavefunctions for the linear system (11).

One always has a freedom to choose the dual wavefunction (or, in other words, the freedom to choose the weight function $\tilde{\rho}(\lambda)$), keeping the rotation coefficients invariant. This freedom is described by the Combescure symmetry transformation between the solutions of the DZM system of equations

$$(\tilde{f}'_i(\boldsymbol{x}))^{-1}\partial_i \tilde{f}'_j(\boldsymbol{x}) = \tilde{f}_i(\boldsymbol{x})^{-1}\partial_i \tilde{f}_j(\boldsymbol{x}). \tag{19}$$

The equations (19) just literally reflect the invariance of the rotation coefficients.

Similarly for the dual DZM system

$$(\partial_i f'_j(\boldsymbol{x}))(f'_i(\boldsymbol{x}))^{-1} = (\partial_i f_j(\boldsymbol{x})) f_i(\boldsymbol{x})^{-1}. \tag{20}$$

In fact, the function Φ is a wavefunction for two linear problems (with different potentials), corresponding to the DZM system and the dual DZM system. A general Combescure transformation changes solutions for both the original system and the dual system (i.e. both functions $\rho(\lambda)$, $\tilde{\rho}(\mu)$). It is also possible to consider two *special* subgroups of the Combescure symmetry transformations group. These two subgroups correspond to the change of only one weight function, $\rho(\lambda)$ or $\tilde{\rho}(\mu)$; we will call transformations of this type right- or left-Combescure transformations, respectively. The invariants for the right (left) Combescure transformations are the solutions of the dual (or the original) DZM system. Another form of these invariants is

$$(\tilde{f}_i'(\boldsymbol{x}))^{-1}\partial_i\Phi'(\boldsymbol{x}) = \tilde{f}_i(\boldsymbol{x})^{-1}\partial_i\Phi(\boldsymbol{x}) \tag{21}$$

$$(\partial_i\Phi'(\boldsymbol{x}))(f_i'(\boldsymbol{x}))^{-1} = (\partial_i\Phi(\boldsymbol{x}))f_i(\boldsymbol{x})^{1}. \tag{22}$$

These equations are important for the connection with the hydrodynamical-type systems [14].

4. The KP–mKP hierarchy

The KP–mKP hierarchy is generated by

$$g(\boldsymbol{x},\lambda) = \exp\left(\sum_{i=1}^{\infty} x_i\lambda^i\right) \tag{23}$$

where $\boldsymbol{x}$ is the set of all the variables x_i, $1 \leqslant i < \infty$. G is a unit disc in this case. Let us take

$$g_1g_2^{-1} = \exp\left(\sum_{i=1}^{\infty}(x_i - x_i')\lambda^{-i}\right) = \exp\left(-\sum_{i=1}^{\infty}\frac{\epsilon^i}{i\lambda^i}\right) = \left(1-\frac{\epsilon}{\lambda}\right).$$

Substituting this function in the Hirota bilinear identity (1), we get

$$\left(1-\frac{\epsilon}{\mu}\right)\chi(\lambda,\mu,\boldsymbol{x}') - \left(1-\frac{\epsilon}{\lambda}\right)\chi(\lambda,\mu,\boldsymbol{x}) = \epsilon\chi(\lambda,0,\boldsymbol{x}')\chi(0,\mu,\boldsymbol{x})$$

$$x_i' - x_i = \frac{1}{i}\epsilon^i \tag{24}$$

or, in terms of the function $\psi(\lambda,\mu)$

$$\psi(\lambda,\mu,\boldsymbol{x}') - \psi(\lambda,\mu,\boldsymbol{x}) = \epsilon\psi(\lambda,0,\boldsymbol{x}')\psi(0,\mu,\boldsymbol{x}) \qquad x_i' - x_i = \frac{1}{i}\epsilon^i. \tag{25}$$

This equation is a finite form of the whole KP–mKP hierarchy. Indeed, the expansion of this relation over ϵ generates the KP–mKP hierarchies (and dual hierarchies) and linear problems for them.

Let us take the first three equations given by the expansion of (25) over ϵ

$$\epsilon: \qquad \psi(\lambda,\mu,\boldsymbol{x})_x = \psi(\lambda,0,\boldsymbol{x})\psi(0,\mu,\boldsymbol{x}) \tag{26}$$

$$\epsilon^2: \qquad \psi(\lambda,\mu,\boldsymbol{x})_y = \psi(\lambda,0,\boldsymbol{x})_x\psi(0,\mu,\boldsymbol{x})\psi(\lambda,0,\boldsymbol{x})\psi(0,\mu,\boldsymbol{x})_x \tag{27}$$

$$\begin{aligned}\epsilon^3: \qquad \psi(\lambda,\mu,\boldsymbol{x})_t &= \tfrac{1}{4}\psi(\lambda,\mu,\boldsymbol{x})_{xxx} - \tfrac{3}{4}\psi(\lambda,0,\boldsymbol{x})_x\psi(0,\mu,\boldsymbol{x})_x \\ &\quad + \tfrac{3}{4}(\psi(\lambda,0,\boldsymbol{x})_y\psi(0,\mu,\boldsymbol{x}) - \psi(\lambda,0,\boldsymbol{x})\psi(0,\mu,\boldsymbol{x})_y) \end{aligned} \tag{28}$$

$$x = x_1 \qquad y = x_2 \qquad t = x_3.$$

In the order ϵ^2 the equation (25) gives rise equivalently to the equations

$$\psi(\lambda,\mu,\boldsymbol{x})_y - \psi(\lambda,\mu,\boldsymbol{x})_{xx} = 2\psi(\lambda,0,\boldsymbol{x})\psi(0,\mu,\boldsymbol{x})_x \tag{29}$$

$$\psi(\lambda,\mu,\boldsymbol{x})_y + \psi(\lambda,\mu,\boldsymbol{x})_{xx} = 2\psi(\lambda,0,\boldsymbol{x})_x\psi(0,\mu,\boldsymbol{x}). \tag{30}$$

Evaluating the first equation at $\mu = 0$, the second at $\lambda = 0$ and integrating them with the weight functions $\rho(\lambda)$ ($\tilde{\rho}(\mu)$), one gets (see (14))

$$f(\boldsymbol{x})_y - f(\boldsymbol{x})_{xx} = u(\boldsymbol{x})f(\boldsymbol{x}) \tag{31}$$

$$\tilde{f}(\boldsymbol{x})_y + \tilde{f}(\boldsymbol{x})_{xx} = u(\boldsymbol{x})\tilde{f}(\boldsymbol{x}) \tag{32}$$

where $u(\boldsymbol{x}) = -2\psi(0, 0, \boldsymbol{x})_x$.

In a similar manner, one obtains from (26)–(28) the equations

$$f_t - f_{xxx} = \tfrac{3}{2}uf_x + \tfrac{3}{4}(u_x + \partial_x^{-1}u_y)f \tag{33}$$

$$\tilde{f}_t - \tilde{f}_{xxx} = \tfrac{3}{2}u\tilde{f}_x + \tfrac{3}{4}(u_x - \partial_x^{-1}u_y)\tilde{f}. \tag{34}$$

Both the linear system (31), (33) for the wavefunction f and the linear system (32), (34) for the wavefunction $\tilde{f}$ give rise to the same KP equation

$$u_t = \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x + \tfrac{3}{4}\partial_x^{-1}u_{yy}. \tag{35}$$

To derive linear problems for the mKP and dual mKP equations, we will integrate equations (26), (29), (30) and (28) with two weight functions $\rho(\lambda)$, $\tilde{\rho}(\mu)$ (see (14))

$$\Phi(\boldsymbol{x})_x = f(\boldsymbol{x})\tilde{f}(\boldsymbol{x}) \tag{36}$$

$$\Phi(\boldsymbol{x})_y - \Phi(\boldsymbol{x})_{xx} = -2f(\boldsymbol{x})\tilde{f}(\boldsymbol{x})_x \tag{37}$$

$$\Phi(\boldsymbol{x})_y + \Phi(\boldsymbol{x})_{xx} = 2f(\boldsymbol{x})_x\tilde{f}(\boldsymbol{x}) \tag{38}$$

$$\Phi(\boldsymbol{x})_t - \Phi(\boldsymbol{x})_{xxx} = -\tfrac{3}{2}f(\boldsymbol{x})_x\tilde{f}(\boldsymbol{x})_x - \tfrac{3}{4}(f(\boldsymbol{x})\tilde{f}(\boldsymbol{x})_y - f(\boldsymbol{x})_y\tilde{f}(\boldsymbol{x})). \tag{39}$$

Using the first equation to exclude f from the second (and $\tilde{f}$ from the third), we obtain

$$\Phi(\boldsymbol{x})_y - \Phi(\boldsymbol{x})_{xx} = v(\boldsymbol{x})\Phi(\boldsymbol{x})_x \tag{40}$$

$$\Phi(\boldsymbol{x})_y + \Phi(\boldsymbol{x})_{xx} = -\tilde{v}(\boldsymbol{x})\Phi(\boldsymbol{x})_x \tag{41}$$

where $v = -2\tilde{f}(\boldsymbol{x})_x/\tilde{f}(\boldsymbol{x})$, $\tilde{v} = 2f(\boldsymbol{x})_x/f(\boldsymbol{x})$.

Similarly, one gets from (28)

$$\Phi(\boldsymbol{x})_t - \Phi(\boldsymbol{x})_{xxx} = \tfrac{3}{2}v(\boldsymbol{x})\Phi(\boldsymbol{x})_{xx} + \tfrac{3}{4}(v_x + v^2 + \partial_x^{-1}v_y)\Phi_x \tag{42}$$

$$\Phi(\boldsymbol{x})_t - \Phi(\boldsymbol{x})_{xxx} = \tfrac{3}{2}\tilde{v}(\boldsymbol{x})\Phi(\boldsymbol{x})_{xx} + \tfrac{3}{4}(\tilde{v}_x + v^2 - \partial_x^{-1}\tilde{v}_y)\Phi_x. \tag{43}$$

The system (40) and (42) gives rise to the mKP equation

$$v_t = v_{xxx} + \tfrac{3}{4}v^2v_x + 3v_x\partial_x^1v_y + 3\partial_x^{-1}v_{yy} \tag{44}$$

while the system (41) and (43) leads to the dual mKP equation, which is obtained from (44) by the substitution $v \to \tilde{v}$, $t \to -t$, $y \to -y$, $x \to -x$.

So the function Φ is simultaneously a wavefunction for the mKP and dual mKP L-operators with different potentials, defined by the dual KP (KP) wavefunctions.

Using equation (28) and relations (40) and (41), it is possible to obtain an equation for the function $\Phi(\boldsymbol{x})$

$$\Phi_t - \frac{1}{4}\Phi_{xxx} - \frac{3}{8}\frac{\Phi_y^2 - \Phi_{xx}^2}{\Phi_x} + \frac{3}{4}\Phi_x W_y = 0 \qquad W_x = \frac{\Phi_y}{\Phi_x}. \tag{45}$$

This equation first arose in Painleve analysis of the KP equation as a singularity manifold equation [16].

The higher analogues of equations (26)–(28) provide us, with the use of relations (40) and (41), with the higher analogues of equation (45). The compact form of the hierarchy of equations for Φ can be obtained from the basic finite relation (25). Integrating both parts of equation (25) with the weights $\rho(\lambda)$ and $\tilde{\rho}(\mu)$, one gets

$$\Phi(\boldsymbol{x}') - \Phi(\boldsymbol{x}) - \epsilon f(\boldsymbol{x}')\tilde{f}(\boldsymbol{x}) = 0. \tag{46}$$

Differentiating (46) with respect to x_1, dividing the result by $f(\boldsymbol{x}')\tilde{f}(\boldsymbol{x})$ and using (40) and (41), one gets

$$\frac{\Phi_{x'}(\boldsymbol{x}') - \Phi_x(\boldsymbol{x})}{\Phi(\boldsymbol{x}') - \Phi(\boldsymbol{x})} = -\frac{\Phi_{y'}(\boldsymbol{x}') + \Phi_{x'x'}(\boldsymbol{x}')}{2\Phi_{x'}(\boldsymbol{x}')} - \frac{\Phi_y(\boldsymbol{x})\Phi_{xx}(\boldsymbol{x})}{2\Phi_x(\boldsymbol{x})}. \tag{47}$$

This equation is a compact form of the *singularity manifold equation hierarchy*.

It is also possible to obtain the finite form of the KP hierarchy in terms of the τ-function, though we do not develop a consistent approach to the τ-function in this work; we are planning to do it later. All we need here is the formula connecting the τ-function and the CBA function (it can be found in [11, 17]). In fact, after some technical work, this formula provides us with the *definition* of the τ-function through the CBA function in terms of the closed 1-form, but this result goes beyond the scope of this paper. Our goal here is just to demonstrate that the equations for the τ-function arise from the equations for the CBA function in a very simple and straightforward way. The equations for the τ-functions obtained below are not new, they exactly coincide with the equations given by the Sato approach, but we believe that the derivation of these equations is new and instructive.

Substituting the expression of the function $\chi(\lambda,\mu)$ through the τ-function

$$\chi(\lambda,\mu) = \frac{\tau(g(\nu)\times((\nu-\lambda)/(\nu-\mu)))}{\tau(g(\nu))(\lambda-\mu)} \tag{48}$$

(see e.g. [11, 17]; the τ-function is a functional of the function $g(\nu)$ or, in other words, a function of $\boldsymbol{x}$) into equation (24), one gets

$$\lambda(\mu-\epsilon)\tau(\boldsymbol{x}^{(5)})\tau(\boldsymbol{x}^{(0)}) + \mu(\epsilon-\lambda)\tau(\boldsymbol{x}^{(4)})\tau(\boldsymbol{x}^{(1)}) + \epsilon(\lambda-\mu)\tau(\boldsymbol{x}^{(3)})\tau(\boldsymbol{x}^{(2)}) = 0 \tag{49}$$

$$x_i^{(5)} - x_i^{(4)} = x_i^{(1)} - x_i^{(0)} = \frac{1}{i}\epsilon^i$$

$$x_i^{(3)} - x_i^{(1)} = x_i^{(4)} - x_i^{(2)} = -\frac{1}{i}\lambda^i$$

$$x_i^{(5)} - x_i^{(3)} = x_i^{(2)} - x_i^{(0)} = \frac{1}{i}\mu^i.$$

The expansion of (49) in ϵ, λ, μ gives the KP hierarchy in the form of Hirota bilinear equations.

Equation (49) is equivalent to that of the addition formulae for the τ-function found in [1].

5. Combescure transformations for the KP–mKP hierarchy

Let us now consider the symmetries of the equations derived above.

Since $\rho(\lambda)$ and $\tilde\rho(\mu)$ are arbitrary functions, equation (45) and the hierarchy (47) possess the symmetry transformation $\Phi(\rho(\lambda),\tilde\rho(\mu)) \to \Phi' = \Phi(\rho'(\lambda),\tilde\rho'(\mu))$. In the context of integrable systems this is exactly the transformation that gives rise to the geometrical Combescure transformation for the DZM system, so we call this transformation *a Combescure symmetry transformation*, in this case without any reference to geometry.

The Combescure transformation can be characterized in terms of the corresponding invariants. The simplest of these invariants for the mKP equation is just the potential of the KP equation L-operator expressed through the wavefunction

$$u = \frac{f(\boldsymbol{x})_y - f(\boldsymbol{x})_{xx}}{f(\boldsymbol{x})} \tag{50}$$

$$u = \frac{\tilde f(\boldsymbol{x})_y - \tilde f(\boldsymbol{x})_{xx}}{\tilde f(\boldsymbol{x})} \tag{51}$$

or, in terms of the solution for the mKP (dual mKP) equation

$$v'_y + v'_{xx} - \tfrac{1}{2}((v')^2)_x = v_y + v_{xx} - \tfrac{1}{2}(v^2)_x \tag{52}$$

$$\tilde v'_y - \tilde v'_{xx} - \tfrac{1}{2}((\tilde v')^2)_x = \tilde v_y - \tilde v_{xx} - \tfrac{1}{2}(\tilde v^2)_x. \tag{53}$$

The solutions of the mKP equations are transformed only by a subgroup of the Combescure symmetry group corresponding to the change of the weight function $\tilde{\rho}(\mu)$ (left subgroup) and they are invariant under the action of the subgroup corresponding to $\rho(\lambda)$ (vice versa for the dual mKP).

All the hierarchy of the Combescure transformation invariants is given by the expansion over ϵ near the point $\boldsymbol{x}$ of the relation (25) rewritten in the form

$$\frac{\partial}{\partial\epsilon}\left(\frac{\tilde{f}(\boldsymbol{x}')-\tilde{f}(\boldsymbol{x})}{\epsilon\tilde{f}(\boldsymbol{x})}\right)=-\frac{1}{2}\frac{\partial}{\partial\epsilon}\partial_{x'}^{-1}u(\boldsymbol{x}') \qquad x_i'-x_i=\frac{1}{i}\epsilon^i \tag{54}$$

$$\frac{\partial}{\partial\epsilon}\left(\frac{f(\boldsymbol{x})-f(\boldsymbol{x}')}{\epsilon f(\boldsymbol{x})}\right)=\frac{1}{2}\frac{\partial}{\partial\epsilon}\partial_{x'}^{-1}u(\boldsymbol{x}') \qquad x_i'-x_i=-\frac{1}{i}\epsilon^i. \tag{55}$$

The expansion of the left part of these relations gives the Combescure transformation invariants in terms of the wavefunctions $\tilde{f}$, f. To express them in terms of mKP equation (dual mKP equation) solution, one should use the formulae

$$v=-2\frac{\tilde{f}_x}{\tilde{f}} \qquad \tilde{f}=\exp\left(-\frac{1}{2}\partial_x^1 v\right) \tag{56}$$

$$\tilde{v}=2\frac{f_x}{f} \qquad f=\exp\left(\frac{1}{2}\partial_x^{-1}\tilde{v}\right). \tag{57}$$

It is also possible to consider special Combescure transformations keeping invariant the KP equation (dual KP equation) wavefunctions (i.e. solutions for the dual mKP (mKP) equations). The first invariants of this type are

$$\frac{\Phi_x'(\boldsymbol{x})}{\tilde{f}'(\boldsymbol{x})}=\frac{\Phi_x(\boldsymbol{x})}{\tilde{f}(\boldsymbol{x})} \tag{58}$$

$$\frac{\Phi_x'(\boldsymbol{x})}{f'(\boldsymbol{x})}=\frac{\Phi_x(\boldsymbol{x})}{f(\boldsymbol{x})}. \tag{59}$$

All the hierarchy of the invariants of this type is generated by the expansion of the left part of the following relations over ϵ

$$\left(\frac{\Phi(\boldsymbol{x}')-\Phi(\boldsymbol{x})}{\tilde{f}(\boldsymbol{x})}\right)=\epsilon f(\boldsymbol{x}') \qquad x_i'-x_i=\frac{1}{i}\epsilon^i \tag{60}$$

$$\left(\frac{\Phi(\boldsymbol{x})-\Phi(\boldsymbol{x}')}{f(\boldsymbol{x})}\right)=\epsilon \tilde{f}(\boldsymbol{x}') \qquad x_i'-x_i=-\frac{1}{i}\epsilon^i. \tag{61}$$

Now let us consider equation (45) and all the hierarchy given by the relation (47). This equation admits the Combescure group of symmetry transformations $\Phi(\rho(\lambda),\tilde{\rho}(\mu))\rightarrow\Phi'=\Phi(\rho'(\lambda),\tilde{\rho}'(\mu))$ consisting of two subgroups (right and left Combescure transformations). These subgroups have the following invariants

$$v=\frac{\Phi_y-\Phi_{xx}}{\Phi_x} \tag{62}$$

and

$$\tilde{v}=\frac{\Phi_y+\Phi_{xx}}{\Phi_x}. \tag{63}$$

From (40) and (41) it follows that they just obey the mKP and dual mKP equations, respectively. The invariant for the full Combescure transformation can be obtained by the substitution of the expression for v via Φ (62) to the formula (52). It reads

$$u=\partial_x^1\left(\frac{\Phi_y}{\Phi_x}\right)_y-\frac{\Phi_{xxx}}{\Phi_x}+\frac{\Phi_{xx}^2-\Phi_y^2}{2\Phi_x^2}. \tag{64}$$

From (31), (32) and (50)–(53) it follows that u solves the KP equation.

So there is an interesting connection between equation (45), mKP–dual-mKP equations and the KP equation. Equation (45) is the unifying equation. It possesses a Combescure symmetry transformations group. After the factorization of equation (45) with respect to one of the subgroups (right or left), one gets the mKP or dual-mKP equation in terms of the invariants for the subgroup (62) and (63). The factorization of equation (45) with respect to the full Combescure transformations group gives rise to the KP equation in terms of the invariant of group (64).

In other words, the invariant of equation (45) under the full Combescure group is described by the KP equation, while the invariants under the action of its right and left subgroups are described by the mKP or dual-mKP equations.

Thus the generalized hierarchy (47) plays a central role in the theory of the KP and mKP hierarchies.

Using the results of the paper [11], it is possible to get the formulae for the Darboux-type transformation for (45) in terms of its special solution $\psi(\lambda, \mu)$. Indeed, a Darboux-type transformation corresponds to

$$g_{\mathrm{d}} = \frac{\nu - b}{\nu - a} \qquad a, b \in \boldsymbol{D} \tag{65}$$

(in fact a, b may also belong to regions not connected with $\boldsymbol{D}$, this case requires some additional definitions). The action of g_{d} (65) on the function $\chi(\lambda, \mu; g)$ is given by the formula (see [11])

$$\chi(\lambda, \mu; g \times g_{\mathrm{d}}) = g_{\mathrm{d}}^{-1}(\lambda) g_{\mathrm{d}}(\mu) \frac{\det\begin{pmatrix} \chi(\lambda, \mu; g) & \chi(\lambda, a; g) \\ \chi(b, \mu; g) & \chi(b, a; g) \end{pmatrix}}{\chi(b, a; g)}. \tag{66}$$

In terms of the function $\psi(\lambda, \mu)$ we get

$$\psi(\lambda, \mu; g \times g_{\mathrm{d}}) = g_{\mathrm{d}}^{-1}(\lambda) g_{\mathrm{d}}(\mu) g(\lambda) g^{-1}(\mu) \times \frac{\det\begin{pmatrix} g^{-1}(\lambda)\psi(\lambda, \mu; g) g(\mu) & g^{-1}(\lambda)\psi(\lambda, a; g) g(a) \\ g^{-1}(b)\psi(b, \mu; g) g(\mu) & g^{-1}(b)\psi(b, a; g) g(a) \end{pmatrix}}{g^{-1}(b)\psi(b, a; g) g(a)} \tag{67}$$

where g is given by (23)

$$g(\boldsymbol{x}, \lambda) = \exp\left(\sum_{i=1}^{\infty} x_i \lambda^i\right).$$

The formula (67) determines a Darboux-type transformation for equation (45) in terms of the function $\psi(\lambda, \mu)$. We note that the functions $\psi(\lambda, a; g)$, $\psi(b, \mu; g)$ and $\psi(b, a; g)$ are connected with the function $\psi(\lambda, \mu; g)$ by the *Combescure transformation* (left, right and their combination). So formula (67) demonstrates an intriguing connection between the Darboux and the Combescure transformations for equation (45).

6. Davey–Stewartson—modified Davey–Stewartson hierarchy. The Ishimori equation

Now we will consider the two-component extension of the KP–mKP hierarchy. We take a set of two identical unit discs with the centre at $\lambda = 0$ $\boldsymbol{D}_+$, $\boldsymbol{D}_-$ as G. The functions $K_+(\lambda)$, $K_-(\lambda)$ are chosen in the form

$$K_+(\lambda) = \lambda^{-1} \ (\lambda \in \boldsymbol{D}_+) \qquad K_-(\lambda) = \lambda^{-1} \ (\lambda \in \boldsymbol{D}_-)$$
$$K_+(\lambda) = 0 \ (\lambda \in \boldsymbol{D}_-) \qquad K_-(\lambda) = 0 \ (\lambda \in \boldsymbol{D}_+).$$

The DS–mDS hierarchy is generated by

$$g(\boldsymbol{x},\lambda)=\exp\left(\sum_{i=1}^{\infty}(x_i^+K_+^i+x_i^-K_-^i)\right).$$

Let us take

$$g_1g_2^{-1}=\exp\left(\sum_{i=1}^{\infty}\left(\frac{\epsilon_+^i}{i}K_+^i+\frac{\epsilon_-^i}{i}K_-^i\right)\right)=(1-\epsilon_+K_+)(1-\epsilon_-K_-).$$

Substituting this function to the Hirota bilinear identity (1), we get

$$\psi(\lambda,\mu,\boldsymbol{x}')-\psi(\lambda,\mu,\boldsymbol{x})=\epsilon_+\psi(\lambda,0_+,\boldsymbol{x}')\psi(0_+,\mu,\boldsymbol{x})+\epsilon_-\psi(\lambda,0_-,\boldsymbol{x}')\psi(0_-,\mu,\boldsymbol{x}) \tag{68}$$
$$(x_i^+)'-x_i^+=\frac{1}{i}\epsilon_+^i\qquad (x_i^-)'-x_i^-=\frac{1}{i}\epsilon_-^i.$$

The expansion of this relation over ϵ_+, ϵ_- generates the DS–mDS hierarchies (and dual hierarchies) and linear problems for them.

The DS equation in the usual form is written in terms of the variables $\xi=\frac{1}{2}(x+y)=x_1^+$, $\eta=\frac{1}{2}(y-x)=x_1^-$, $t=-\frac{1}{2}\mathrm{i}(x_2^+-x_2^-)$. The DS hierarchy in the form (69) also incorporates the modified Veselov–Novikov hierarchy.

In the standard DS coordinates one gets from (69)

$$\psi(\lambda,\mu,\boldsymbol{x})_\xi=\psi(\lambda,0_+,\boldsymbol{x})\psi(0_+,\mu,\boldsymbol{x}) \tag{69}$$
$$\psi(\lambda,\mu,\boldsymbol{x})_\eta=\psi(\lambda,0_-,\boldsymbol{x})\psi(0_-,\mu,\boldsymbol{x}) \tag{70}$$
$$\begin{aligned}\mathrm{i}\psi(\lambda,\mu,\boldsymbol{x})_t=\tfrac{1}{2}(&\psi(\lambda,0_+,\boldsymbol{x})_\xi\psi(0_+,\mu,\boldsymbol{x})-\psi(\lambda,0_+,\boldsymbol{x})\psi(0_+,\mu,\boldsymbol{x})_\xi\\&-\psi(\lambda,0_-,\boldsymbol{x})_\eta\psi(0_-,\mu,\boldsymbol{x})+\psi(\lambda,0_-,\boldsymbol{x})\psi(0_-,\mu,\boldsymbol{x})_\eta).\end{aligned} \tag{71}$$

Just as in the DZM system case, from (69) and (70) one obtains the DS and dual DS spatial linear problems

$$\begin{aligned}&\partial_\eta f_-=uf_+\qquad \partial_\eta\tilde{f}_-=v\tilde{f}_+\\&\partial_\xi f_+=vf_-\qquad \partial_\xi\tilde{f}_+=u\tilde{f}_-.\end{aligned} \tag{72}$$

Here $v=\psi(0_-,0_+)$, $u=\psi(0_+,0_-)$.

Similarly to the KP equation case, (71) gives a time linear problem for the DS equation

$$\begin{aligned}&if_{+t}-\tfrac{1}{2}f_{+\xi\xi}+\tfrac{1}{2}f_{+\eta\eta}=(\partial_\eta^{-1}(uv)_\xi)f_+-v_\eta f_-\\&if_{-t}-\tfrac{1}{2}f_{-\xi\xi}+\tfrac{1}{2}f_{+\eta\eta}=-(\partial_\xi^{-1}(uv)_\eta)f_-+u_\xi f_+.\end{aligned} \tag{73}$$

The compatibility condition for (72) and (73) gives the DS equation (in fact it is even easier to obtain it directly from (69)–(71))

$$\begin{aligned}&iv_t-\tfrac{1}{2}v_{\xi\xi}-\tfrac{1}{2}v_{\eta\eta}=-((\partial_\xi^{-1}(uv)_\eta)+(\partial_\eta^{-1}(uv)_\xi))v\\&iu_t+\tfrac{1}{2}u_{\xi\xi}+\tfrac{1}{2}u_{\eta\eta}=((\partial_\eta^{-1}(uv)_\xi)+(\partial_\xi^{-1}(uv)_\eta))u.\end{aligned} \tag{74}$$

The spatial and time linear problems for the mDS–dual mDS case read

$$\Phi_{\eta\xi}=U_\xi\Phi_\eta+V_\eta\Phi_\xi \tag{75}$$
$$\mathrm{i}\Phi_t+\tfrac{1}{2}\Phi_{\eta\eta}-\tfrac{1}{2}\Phi_{\xi\xi}=V_\eta\Phi_\eta-U_\xi\Phi_\xi \tag{76}$$

and

$$\Phi_{\xi\eta}=\tilde{U}_\xi\Phi_\eta+\tilde{V}_\eta\Phi_\xi \tag{77}$$
$$\mathrm{i}\Phi_t-\tfrac{1}{2}\Phi_{\eta\eta}+\tfrac{1}{2}\Phi_{\xi\xi}=-\tilde{V}_\eta\Phi_\eta+\tilde{U}_\xi\Phi_\xi. \tag{78}$$

Here $V = \log \tilde{f}_+$, $U = \log \tilde{f}_-$, $\tilde{V} = \log f_+$, $\tilde{U} = \log f_-$. The compatibility condition for (75) and (76) gives the equations [18]

$$\begin{aligned}\left(\mathrm{i}U_t - \tfrac{1}{2}U_{\xi\xi} - \tfrac{1}{2}U_{\eta\eta} - \tfrac{1}{2}U_\xi^2 - \tfrac{1}{2}U_\eta^2 + U_\eta V_\eta\right)_\eta + \left(U_\eta V_\xi\right)_\xi &= 0\\ \left(\mathrm{i}V_t + \tfrac{1}{2}V_{\xi\xi} + \tfrac{1}{2}V_{\eta\eta} - \tfrac{1}{2}V_\xi^2 - \tfrac{1}{2}V_\eta^2 + U_\xi V_\xi\right)_\xi + \left(U_\eta V_\xi\right)_\eta &= 0\end{aligned} \tag{79}$$

which can be treated as the modified DS equation. This system and its connection with the DS equation has been analysed in [18]. The dual modified DS equation can be obtained from (79) by the substitution $V \to \tilde{V}$, $U \to \tilde{U}$, $t \to -t$, $\xi \to -\xi$, $\eta \to -\eta$.

Solutions for this system are given in terms of dual DS wavefunctions (DS wavefunctions). Thus there is a Combescure transformations group acting on the space of solutions. The simplest Combescure invariants are

$$\frac{\partial_\eta \tilde{f}_-}{\tilde{f}_+} = V_\eta \exp(V - U) \tag{80}$$

$$\frac{\partial_\xi \tilde{f}_+}{\tilde{f}_-} = U_\xi \exp(U - V). \tag{81}$$

The hierarchy of the Combescure transformation invariants is generated by the relations

$$\left(\frac{\tilde{f}_+(\boldsymbol{x}') - \tilde{f}_+(\boldsymbol{x})}{\tilde{f}_-(\boldsymbol{x})}\right) = \epsilon_+ u(\boldsymbol{x}') \tag{82}$$

$$(x_i^+)' - x_i^+ = \frac{1}{i}\epsilon_+^i \qquad x_i^- - (x_i^-)' = 0$$

$$\left(\frac{\tilde{f}_-(\boldsymbol{x}') - \tilde{f}_-(\boldsymbol{x})}{\tilde{f}_+(\boldsymbol{x})}\right) = \epsilon_- v(\boldsymbol{x}') \tag{83}$$

$$(x_i^-)' - x_i^- = \frac{1}{i}\epsilon_-^i \qquad (x_i^+)' - x_i^+ = 0.$$

If one takes a pair of wavefunctions $\tilde{f}_+$, $\tilde{f}_-$ and $\tilde{f}'_+$, $\tilde{f}'_-$, the matrix

$$\Psi = \begin{pmatrix} \tilde{f}_+ & \tilde{f}'_+ \\ \tilde{f}_- & \tilde{f}'_- \end{pmatrix}$$

is connected with the solution of the Ishimori equation by the formula (see e.g. [18])

$$S_1\sigma_1 + S_2\sigma_2 + S_3\sigma_3 = -\Psi^1\sigma_3\Psi$$

(for real $\boldsymbol{S}$ some reduction conditions should be satisfied). In principle it could be possible to express Combescure invariants for mDS equation in terms of solution for the Ishimori equation and thus obtain Combescure invariants for the Ishimori equation, but it is unclear in this case whether the Combescure transformation survives under reduction conditions.

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